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TOPOLOGICAL DERIVATIVES FOR ELLIPTIC PROBLEMS ON GRAPHS

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Abstract. We consider elliptic problems on graphs under given loads. We ask the question which graph is best suited to sustain the loads. More precisely, given a cost function we may look at a multiple node of the graph and ask as to whether that node should be resolved into a number of nodes of edge degree 3, in order to decrease the cost. Thus, we are looking into the topological gradient of an elliptic problem on a graph.

Key words: topological derivative, shape optimization, asymptotic analysis, graph theory, singular perturbations, topology optimization

1 Introduction

For a considerable number of important problems the notion of topological derivatives has been introduced, and examples for such gradients have been reported in the literature. The list of problems considered comprises elliptic problems in 2 and 3 dimensions with and without obstacles, the equations of elasticity and the Helmholtz equation. See Sokolowski [10], Amstutz [2], Allaire et.al. [1], Masmoudi et. al. [7], Novotny et.al.[8] and others together with the references therein. Topological derivatives are important in dealing with topology and shape optimization. The reason for this fact is that homeomorphic variations of the domains will not allow for topology changes. Thus if one considers a shape optimization problem and starts with a simply connected set, say, then all admissible variations will produce simply connected sets. If, therefore, an optimal shape would necessitate digging a hole into the domain, then it would not be possible to do this with the kind of domain variations mentioned. Topological gradients are obviously a key ingredient in topology optimization, the boundary between these disciplines becoming increasingly floating.

However, the topological gradient is more a qualitative tool than a quantitative one: it helps to indicate where a hole has to be located. The actual optimization of the domain is then subject to shape-sensitivities.

Topology optimization for graph-like problems has been considered in the engineering literature for a long time. See Rozvany et.al.[9] as an example. Truss optimization has also been the focus of many mathematical papers. However, to best knowledge of the authors such truss problems do not describe flexible systems as they use rod-models instead of flexible beam models, nor do they consider 1-d elasticity models other than their finite element approximations. The method used there typically comes down to selecting rod elements out of a complete graph in order to decrease a given cost (the typical choice being the compliance). We instead aim at graph structures which are locally described by partial differential equations along the edges of the underlying graph. In this paper we confine ourselves with second order equations which are representative of 1-d elasticity. Timoshenko-beam and Euler-Bernoulli beams will be discussed in a forthcoming publication.

Networks carrying dynamics appear in many applications, such as neuronal dynamics, waste-water management, blood flow, micro-flows, gas- and traffic networks and many more. In all these applications the optimization of the topology of the graph is crucial. Thus it appears reasonable to approach this kind of problem with a topological gradient calculus.

To the best knowledge of the authors, topological gradients for partial differential equations on graphs have not been considered within the literature.

The first author has been working on partial differential equations on networked domains during the last 10 years. See the monographs by Lagnese, Leugering and Schmidt [5] and Lagnese and Leugering [6] for further reference on the modeling of such problems. For the sake of self-consistency we introduce the models below.

The paper is organized as follows. In the second section we provide preliminaries on elliptic problems on graphs. The third section is devoted to the Steklov-Poincaré operator on the graph. In the fourth section we develop the asymptotic expansions for the problems on graphs with a hole. The last section will be devoted to asymptotic expansions of the energy and a tracking functional.

2 Preliminaries

We consider a simple graph $(V, E) = G$ in $\mathbf{R}^d$, $d = 2, 3$, with vertices $V = \{v_J | J \in \mathcal{J}\}$ and edges $E = \{e_i | i \in \mathcal{I}\}$. Let $m = |\mathcal{J}|$, $n = \|\mathcal{I}\|$ be the numbers of vertices and edges, respectively. Given a node v_J we define

$$\mathcal{I}_J := \{i \in \mathcal{I} | e_i \text{ is incident at } v_J\}$$

the incidence set, and $d_J = |\mathcal{J}|$ the edge degree of v_J. The set of nodes splits into simple nodes $\mathcal{J}_S$ and multiple nodes $\mathcal{J}_M$ according to $d_J = 1$ and $d_J > 1$, respectively.

On G we consider a function

$$r : G \to \mathbf{R}^{np} := \Pi^{p_i}_{i \in \mathcal{I}},\ p_i \geq 1\, \forall i. \tag{1}$$

The numbers p_i represent the degrees of freedom of the physical model used to describe the behavior of the edge with number i. For instance, $p = 1$ is representative

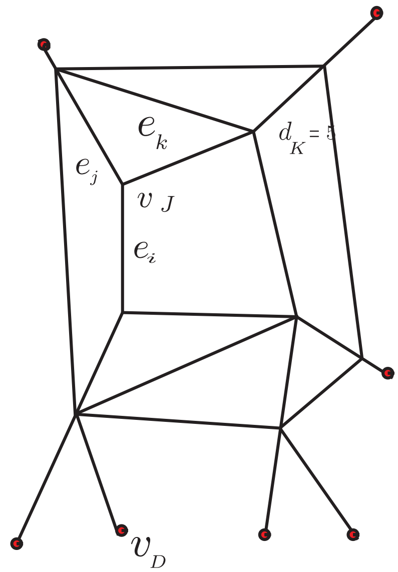

Figure 1: A general graph

of a heat problem, whereas $p = 2, 3$ is used in an elasticity context on graphs in 2 or 3 dimensions. The $p_i's$ may change in the network in principle. However, in this paper we insist on $p_i = p, \forall i$. To keep matters simple, we also take the arcs as straight lines. The more general case, which is of course also interesting in the combination of shape and topology optimization, can also be handled. See Lagnese, Leugering and Schmidt[5] and Lagnese and Leugering [6] for details on the modeling.

Once the function r is understood as being representative of, say, a deformation of the graph, we may localize it to the edges

$$r_i := r|_{e_i} : [\alpha_i, \beta_i] \to \mathbf{R}^p, \ i \in \mathcal{I}, \tag{2}$$

where e_i is parametrized by $x \in [\alpha_i, \beta_i] =: I_i$,$0 \leq \alpha_i < \beta_i$, $\ell_i := \beta_i - \alpha_i$. We introduce the incidence relation

$$d_{iJ} := \begin{cases} 1 & \text{if } e_i[\text{ ends at } v_J \\ -1 & \text{if } e_i \text{ starts at } v_J \end{cases}$$

Accordingly, we define

$$x_{iJ} := \begin{cases} 0 & \text{if } d_{iJ} = -1 \\ \ell_i & \text{if } d_{iJ} = 1 \end{cases}$$

We will use the notation $r_i(v_J)$ instead of $r_i(x_{iJ})$. In order to represent the material considered on the graph, we introduce stiffness matrices

$$K_i := h_i[(1 - \frac{1}{s_i})I + \frac{1}{s_i} e_i e_i^T] \tag{3}$$

where we now use the notation e_i as the normalized vectors along the edge i. Obviously, the longitudinal stiffness is given by h_i, whereas the transverse stiffness is given by $h_i(1 - \frac{1}{s_i})$. This can be related to 1-d analoga of the Lamé parameters. We introduce Dirichlet and Neumann simple nodes

$$\mathcal{J}_D := \{J \in \mathcal{J}_S | r_i(v_D) = 0\}$$

$$\mathcal{J}_N := \{J \in \mathcal{J}_S | d_{iJ} K_i r_i'(v_N) = 0\}$$

Notice that $\mathcal{I}_J, J \in \mathcal{J}_N \cup \mathcal{J}_D$ is a singleton. The basic assumption at a multiple node is that the deformation r is continuous across the joint. In truss design this is not the case, and consequently pin-joints are considered, however on a discrete level. One may consider pin-joints also on the continuous level, as in Lagnese, Leugering and Schmidt[5] and [6]. In this paper we restrict ourselves to 'rigid' joints in the sense that the angles between edges in their reference configuration remain fixed. The continuity is expressed simply as

$$r_i(v_J) = r_j(v_J), \quad i, j \in \mathcal{I}_J,\ J \in \mathcal{J}_M$$

We consider the energy of the system

$$\mathcal{E}_0 := \frac{1}{2} \sum_{i \in \mathcal{I}} \int_0^{\ell_i} K_i r_i' \cdot r_i' + c_i r_i \cdot r_i dx \tag{4}$$

where the primes denote the derivative with respect to the running variable x_i, c_i represents a reaction term or an elastic support.

In order to analyze the problem, we need to introduce a proper energy space

$$\begin{aligned} \mathcal{V} \;:=\; & \{r : G \to \mathbf{R}^{np} | r_i \in H^1(I_i) & (5)\\ & r_i(v_D) = 0,\ i \in \mathcal{I}_D,\ D \in \mathcal{J}_D & (6)\\ & r_i(v_J) = r_j(v_J),\ \forall i, j \in \mathcal{I}_J,\ J \in \mathcal{J}_M\} & (7) \end{aligned}$$

$\mathcal{V}$ is clearly a Hilbert space in

$$\mathcal{H} := L^2(0, \ell_i)^{np} \tag{8}$$

We introduce the bilinear form on $\mathcal{V} \times \mathcal{V}$

$$a(r, \phi) := \sum_{i \in \mathcal{I}} \int_0^{\ell_i} [K_i r_i' \cdot \phi_i' + c_i r_i \cdot \phi_i] dx. \tag{9}$$

Let now distributed and boundary data, $f_i,\ g_J$ be given along the edge e_i and the node v_J, respectively. Then we may consider the following variational problem in $\mathcal{V}$

$$a(r, \phi) = L(\phi),\ \forall \phi \in \mathcal{V} \tag{10}$$

with

$$L(\phi) := \sum_{i \in \mathcal{I}^f} \int_0^{\ell_i} f_i \cdot \phi_i dx + \sum_{J \in \mathcal{J}_N^g} g_J \cdot \phi_{\hat{i}J}(v_J), \tag{11}$$

where $\hat{i}$ indicates that the simple nodes have just one incident edge. For $f_i \in H^1(0, \ell_i)^*$ (in fact, if the edge e_i is incident to a Dirichlet node, an additional boundary condition appears) such that $\max_i |f_i| \le C$ and $\max_J |g_J| \le C$, we may apply the Lax-Milgram-Lemma and hence obtain a unique solution $r \in \mathcal{V}$ of problem (10),(11). The strong version of (10),(11) is obtained by integration by parts and taking variations in $\mathcal{V}$. See Lagnese, Leugering and Schmidt [5],[6] for the details. We obtain the following system:

$$\begin{cases} -K_i r_i'' + c_i r_i = f_i, \ i \in \mathcal{I} \\ r_i(v_D) = 0, \ i \in \mathcal{I}_D, \ D \in\in \mathcal{J}_D \\ d_{iJ} K_i r_i'(v_N) = g_J, \ i \in \mathcal{I}_N, \ N \in \mathcal{J}_N \\ r_i(v_J) = r_j(v_J), \ \forall i,j \in \mathcal{I}_J, \ J \in \mathcal{J}_M \\ \sum_{i \in \mathcal{I}_J} d_{iJ} K_i r_i'(v_J) = 0, \ J \in \mathcal{J}_M \end{cases} \tag{12}$$

where $f_i = 0, \ i \in \mathcal{I} \backslash \mathcal{I}^f$, $g_N = 0, \ J \in \mathcal{J}_N \backslash \mathcal{J}_N^g$. Notice that (12) line 5 is an example of the classical Kirchhoff condition known from electrostatics. Notice also that in the vectorial case $d = p = 2$ we have

$$\begin{cases} r_i = y_i e_i + w_i e_i^\perp \\ K_i r_i = h_i y_i e_i + h_i (1 - \frac{1}{s_i} w_i) e_i^\perp \end{cases} \tag{13}$$

and therefore the transmission conditions (12) lines 4 and 5 are to be understood as vectorial quantities.

3 Steklov-Poincaré operators on graphs

In order to proceed with the introduction of a topological gradient, we consider a multiple node $v_J^0, \ J \in \mathcal{J}_M$. Let the edge degree d_J^0 be greater or equal to three, thus we do not consider a serial junction. Ultimately we would like to cut out a star-subgraph

$$\mathcal{S}^{J^0} := \{e_i | i \in \mathcal{I}_{J^0}\} \subset E, \ (S^{J^0}, v_{J^0}) = G_{J^0} \subset G \tag{14}$$

and connect the adjacent nodes. This we consider as digging a hole into the given graph.

We would like to use Steklov-Poincaré operators in order to decompose the entire graph into a subgraph and the remaining network (the exterior). In order to do this we pick Dirichlet-values at the simple vertices of the subgraph obtained by the 'cuts' and evaluate the corresponding Neumann-data there. This constitutes the Steklov-Poincaré operator. The decomposition method applies to any subgraph. Thus the 'effect' of the subgraph can be represented in the context of the overall problem by the way of the Steklov-Poincaré operator corresponding to the subgraph. In order to be able to handle holes with varying sizes, we consider decomposing the graph into an exterior part and a subgraph containing the node v_{J^0} to be cut out. That node is considered together with its adjacent edges, however with edge-lengths ρ_i. The latter star-graph, in turn, is then cut out of the subgraph. Therefore, we obtain the analogue of a ring-like subgraph which constitutes the Steklov-Poincaré subgrpah. See Figure 2 for a typical general situation and Figure 3 for the exemplary local handling of subgraph removal.

In order to simplify the notation, and in fact without loss of generality, we may consider the subgraph (from which the hole is then subsequently removed) as a star with edge degree $d_J(v_{J^0}) = q$.

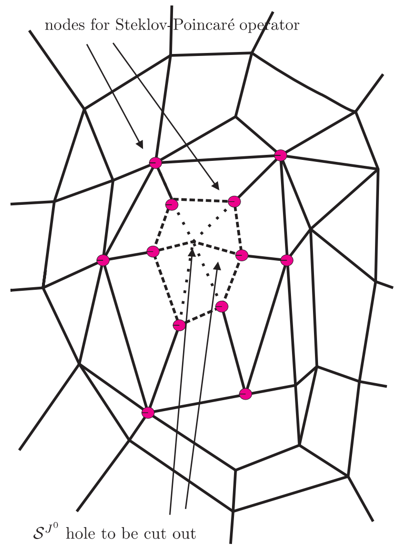

Figure 2: Graph with star-like subgraph to be cut out

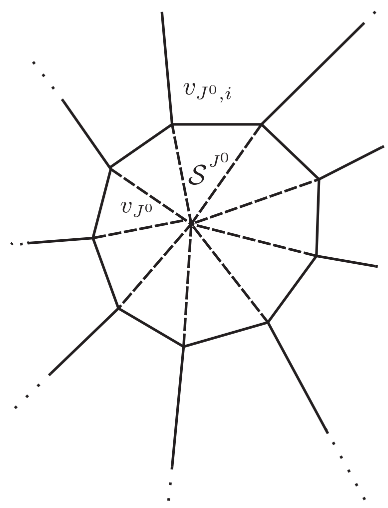

Figure 3: A star-like subgraph

We are led to study the following subproblem

$$\left\{\begin{array}{c} -K_i r_i'' + c_i r_i = f_i,\ i \in \mathcal{I}_{J^0} \\ r_i(v_{J^0,i}) = u_i,\ i \in \mathcal{I}_{J^0} \\ r_i(v_{J^0}) = r_j(v_{J^0}),\ \forall i, j \in \mathcal{I}_{J^0} \\ \sum_{i \in \mathcal{I}_J} d_{iJ} K_i r_i'(v_{J^0}) = 0, \end{array}\right. \tag{15}$$

where $v_{J_i^0} = v_{J^0,i}$ are the nodes adjacent to v_J. See Figure 3

We assume for simplicity that v_{J^0} is an interior node with edge degree q such that its adjacent nodes are not simple. Problem (15) admits a unique solution $r^{i,0},\ i = 1, \dots q$. We consider the Dirichlet-Neumann-map or the Steklov-Poincaré-map

$$\begin{cases} S_{J^0} : \mathbf{R}^{qp} \to \mathbf{R}^{qp} \\ S^i_{J^0} := d_{iJ} K_i r'_{i,0}(v_{J^0,i}),\ i \in \mathcal{I}_{J^0} \end{cases} \tag{16}$$

In order to simplify notation we may assume that the nodes $v_{J^0,i}$, which are the nodes incident at v_{J^0}, have edge degree ≥ 3 in G, such that after cutting the corresponding edges out of G they are still multiple, but now in $G \setminus G_{J^0}$.

The relevance of the Steklov-Poincaré map in this context becomes apparent when we consider the overall problem. Indeed, we solve the problem (15) generate the Neumann data (16) and integrate those into the system with the hole as follows

$$\begin{cases} -K_i r''_i + c_i r_i = f_i,\ \in i \in \mathcal{I} \\ r_i(v_D) = 0,\ i \in \mathcal{I}_D,\ D \in \mathcal{J}_D \\ d_{iJ} K_i r'_i(v_N) = g_J,\ i \in \mathcal{I}_N,\ N \in \mathcal{J}_N \\ r_i(v_J) = r_j(v_J),\ \forall i,j \in \mathcal{I}_J,\ J \in \mathcal{J}_M \setminus \mathcal{J}^0_S \\ \displaystyle\sum_{i \in \mathcal{I}_J} d_{iJ} K_i r'_i(v_J) = 0,\ J \in \mathcal{J}_M \setminus \mathcal{J}^0_S \\ r_k(v_J) = r_\ell(v_J) = r_i(v_{J^0,i})\ \forall k,\ell \in \mathcal{I}_{J^0_S},\ i \in \mathcal{I}_{J^0} \\ \displaystyle\sum_{j \in \mathcal{I}_{J^0_i}} d_{j,J^0_i} K_j r'_j(v_{J^0,i}) + S^i_{J^0}(r_i(v_{J^0,i})) = 0,\ i \in \mathcal{I}_{J^0} \end{cases}, \tag{17}$$

where $S^i_{J^0}(r_{\hat{j}}(v_{J^0,i}))_i$ is the Steklov-Poincaré-map applied to the nodal data at $v_{J^0,i}$. The problem (17) is equivalent to the original problem (12). Obviously, there is nothing special about cutting out a star-subgraph. One may as well cut out any subgraph, solve the corresponding Steklov-Poincaré problem, and read it into the graph problem with the 'hole'. The procedure itself is also completely natural in most of the known domain decomposition techniques. See Lagnese and Leugering [6] for domain decompostion techniques in the context of optimal control problems on networked domains.

4 Stars with a hole

We consider a star-graph G_{J^0} with q edges and center at the node v_{J^0}. As has been seen in the previous section, we may consider this problem completely independent of the original graph. In particular, we may without loss of generality, assume that the edges e_i stretch from the center to the simple boundary nodes, which we will label from 1 to q. By this assumption we consider the multiple node at the center as being reached at $x = 0$ for all outgoing edges. Thus, the data u_i are picked up at the ends $x = \ell_i$.

$$\begin{cases} -K_i r''_i + c_i r_i = f_i,\ i \in \mathcal{I} \\ r_i(\ell_i) = u_i,\ i = 1, \ldots, q \\ r_i(0) = r_j(0),\ \forall i,j = 1, \ldots, q \\ \displaystyle\sum_{i=1}^{q} K_i r'_i(0) = 0. \end{cases} \tag{18}$$

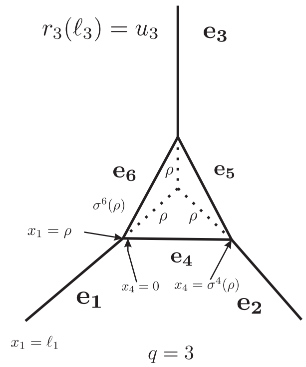

Figure 4: Cutting a hole into star-like subgraph

We are going to cut out the center and connect the corresponding cut-nodes via a circuit as seen in the Figure 4. In general we have numbers $\rho_i \in [0, \ell_i),\ i = 1, \ldots, q$ which are taken to be the lengths of the edges that are cut out. Thus the remaining edges have lengths $\ell_i - \rho_i$. At $x = \rho_i$ we create a new multiple node v_i. We connect these nodes by edges $e_{q+i},\ i = 1, \ldots, q$ with lengths $\sigma^i(\rho_i)$. After that, these nodes receive a new edge degree. In this paper we assume that all these nodes have the same edge degree $d_i = 3$. More complicated cutting procedures can be introduced, but obscure the ideas of this first paper on topological derivatives of graph problems.

The problem we have to solve is the following

$$\left\{\begin{array}{c} -K_i r_i'' + c_i r_i = f_i,\ i \in \mathcal{I} \\ r_i(\ell_i) = u_i,\ \imath = 1, \ldots, q \\ r_i(\rho_i) = r_{q+i}(0) = r_{q+1-i}(\sigma^i(\rho_i)),\ \forall i = 2, \ldots, q \\ r_1(\rho_1) = r_{q+1}(0) = r_{2q}(\sigma^{2q}(\rho_{2q})), \\ -K_i r_i'(\rho_i) - K_{q+i} r_{q+i}'(0) + K_{q+i-1} r_{q+i-1}'(\sigma^{q+i-1}(\rho_{q+i-1})) = 0,\ i = 2, \ldots, q \\ -K_1 r_1'(\rho_1) - K_{q+1} r_{q+1}'(0) + K_{2q} r_{2q}'(\sigma^{2q}(\rho_{2q})) = 0. \end{array}\right. \tag{19}$$

We proceed to derive the solutions to (18) and(19), respectively. To this end we look at

$$-K_i r_i'' + c_i r_i = f_i \Leftrightarrow r_i'' + c_i K_i^{-1} r_i = K_i^{-1} f_i$$

and define $A_i := c_i K_i^{-1},\ F_i := \frac{1}{c_i} A_i f_i$. The general solution of the homogeneous equation ($f_i = 0$) is given by

$$r_i^H(x) = \sinh(A_i^{\frac{1}{2}} x) a_i + \cosh(A_i^{\frac{1}{2}} x) b_i \tag{20}$$

The inhomogeneous equation is then solved by variation of constants as follows

$$r_i^I(x) = A_i^{-\frac{1}{2}} \int_0^x \sinh(A_i^{\frac{1}{2}}(x-s) F_i(s) ds. \tag{21}$$

We will treat the case $f_i = 0$ only. The general case is then a matter of additional but straightforward calculus.

Lemma 4.1 *The solution r to problem* (18) *with $f_i = 0,\ i = 1, \dots, q$ is given by*

$$r_i(x) = \sinh((c_i K_i^{-1})^{\frac{1}{2}}(x))a_i + \cosh((c_i K_i^{-1})^{\frac{1}{2}}(x))b \tag{22}$$

with the coefficient-vectors a_i, b given by

$$\begin{aligned} a_i \quad &= \quad \sinh(A_i^{\frac{1}{2}}\ell_i)^{-1}(u_i - \cosh(A_i^{\frac{1}{2}}) \\ &\qquad \cdot(\sum_{i=1}^{q} \frac{1}{c_i} A_i^{-\frac{1}{2}} \coth A_i^{\frac{1}{2}}\ell_i)^{-1} \sum_{i=1}^{q} \frac{1}{c_i} A_i^{-\frac{1}{2}} \sinh(A_i^{\frac{1}{2}}\ell_i)^{-1} u_i \end{aligned} \tag{23}$$

$$b = (\sum_{i=1}^{q} \frac{1}{c_i} A_i^{-\frac{1}{2}} \coth(A_i^{\frac{1}{2}}))^{-1} \sum_{i=1}^{q} \frac{1}{c_i} A_i^{-\frac{1}{2}} \sinh(A_i^{\frac{1}{2}}\ell_i)^{-1} u_i \tag{24}$$

The Stekov-Poincaré map is given by

$$S^i_{J^0}(u) = A_i^{\frac{1}{2}}(\cosh(A_i^{\frac{1}{2}}\ell_i)a_i + \sinh(A_i^{\frac{1}{2}}\ell_i)b) \tag{25}$$

with a_i, b according to (23),(24).

The situation appears to be much more simple in case all material parameters and geometrical data are equal.

$$c_i = 1,\ K_i = Id = A^{\frac{1}{2}},\ \ell_i = \ell,\ f_i = 0, \quad i = 1, \dots q \tag{26}$$

Example 4.1 *Let assumption* (26) *hold true. Then the solution r to* (18) *is given by*

$$\begin{aligned} r_i(x) \quad &= \quad \frac{1}{\sinh(\ell)} \sinh(x)(u_i - \frac{1}{q}\sum_{j=1}^{q} u_j) \\ &\qquad + \frac{1}{\cosh(\ell)} \cosh(x) \frac{1}{q} \sum_{i=1}^{q} u_i \end{aligned} \tag{27}$$

The Steklov-Poincaré map is given by

$$S^i(u)_{J^0} = \coth(\ell)(u_i - \frac{1}{q}\sum_{j=1}^{q} u_j) + \tanh(\ell)\frac{1}{q}\sum_{j=1}^{q} u_j \tag{28}$$

We proceed to problem (19). Again, we will treat the general case first and will then restrict to assumption (26) in order to better reveal the underlying structure.

We introduce the ansatz for the solution as follows

$$r_i^\rho(x) := \sinh(A_i^{\frac{1}{2}}x)a_i^\rho + \cosh(A_i^{\frac{1}{2}}x)b_i^\rho \tag{29}$$

From the Dirichlet conditions in $(19)_2$ we infer

$$r_i^\rho(\ell_i) = \sinh(A_i^{\frac{1}{2}}(\ell_i))a_i^\rho + \cosh(A_i^{\frac{1}{2}}(\ell_i))b_i^\rho = u_i,\ i = 1, \dots, q. \tag{30}$$

From the continuity requirement in $(19)_{3,4}$ we obtain

$$\begin{aligned} r_i^\rho(\rho_i) &= \sinh(A_i^{\frac{1}{2}}\rho_i)a_i^\rho + \cosh(A_i^{\frac{1}{2}}\rho_i)b_i^\rho = r_{q+i}^\rho(0) = b_{q+i}^\rho \\ &= r_{q+i-1}^\rho(\sigma^{q+i-1}(\rho_{q+i-1})),\ i = 2,\dots q \end{aligned} \tag{31}$$

$$\begin{aligned} r_1^\rho(\rho_1) &= \sinh(A_1^{\frac{1}{2}}\rho_1)a_1^\rho + \cosh(A_1^{\frac{1}{2}}\rho_1)b_1^\rho \\ &= r_{q+1}^\rho(0) = b_{q+1}^\rho = r_{2q}^\rho(\sigma^{2q}(\rho_{2q})) \end{aligned} \tag{32}$$

The Kirchhoff conditions in (19) result in

$$\begin{aligned} &-\frac{1}{c_i}A_i^{-\frac{1}{2}}[\cosh(A_i^{\frac{1}{2}}\rho_i)a_i^\rho + \sinh(A_i^{\frac{1}{2}}\rho_i)b_i^\rho] - \frac{1}{c_{q+i}}A_{q+i}^{-\frac{1}{2}}a_{q+i}^\rho \\ &\quad + \frac{1}{c_{q+i-1}}A_{q+i-1}^{-\frac{1}{2}}[\cosh(A_{q+i-1}^{\frac{1}{2}}(\sigma^{q+i-1}(\rho_{q+i-1})))a_{q+i-1}^\rho \\ &\quad\quad + \sinh(A_{q+i-1}^{\frac{1}{2}}(\sigma^{q+i-1}(\rho_{q+i-1})))b_{q+i-1}^\rho] = 0,\ i = 2,\dots,q \end{aligned} \tag{33}$$

$$\begin{aligned} &-\frac{1}{c_1}A_1^{-\frac{1}{2}}[\cosh(A_1^{\frac{1}{2}}\rho_1)a_1^\rho + \sinh(A_1^{\frac{1}{2}}\rho_1)b_1^\rho] - \frac{1}{c_{q+1}}A_{q+1}^{-\frac{1}{2}}a_{q+1}^\rho \\ &\quad + \frac{1}{c_{2q}}A_{2q}^{-\frac{1}{2}}[\cosh(A_{2q}^{\frac{1}{2}}(\sigma^{2q}(\rho_{2q})))a_{2q}^\rho \\ &\quad\quad + \sinh(A_{2q}^{\frac{1}{2}}(\sigma^{2q}(\rho_{2q})))b_{2q}^\rho] = 0, \end{aligned} \tag{34}$$

This set of equations ((30)-(34))constitutes $4q$ conditions on the $4q$ unknowns $a_i^\rho,\ b_i^\rho,\ i = 1,\dots,2q$. The problem is as to whether there is an asymptotic expansion of r_i^ρ in terms of ρ for small $\rho := (\rho_i)_{i=1,\dots,q}$. This problem is a singular perturbation problem. Notice that the graph with $\rho = 0$ is the original star-graph with q edges, while for every $\rho > 0$ (i.e. $\rho_i > 0$), the graph has $2q$ edges and contains exactly one circuit. We may of course also formally start with a star-graph consisting of $2q$ edges with serial joints at $x_i = 0,\ x_{q+i} = \rho_i,\ i = 1,\dots,q$ so that the edges $e_i,\ i = 1,\dots,q$ have length $\ell_i - \rho_i$ to begin with, while the other edges $e_{q+i},\ i = 1,\dots,q$ stretch from the center (at $x_{q+i} = 0$) to the serial nodes at $x_{q+i} = \rho_i$. But still, the perturbation is then singular with respect to the subgraphs spanned by the edges $e_{q+i}, i = 1\dots q$.

Our analysis depends on the expansion of the set of equations (30) to (34) up to second order terms. The asymptotic analysis is based on the expansions of $\sinh(x), \cosh(x)$ on the matrix level. By spectral decomposition we have

$$\sinh(A_i^{\frac{1}{2}}(x))\xi = \sum_{j=1}^p \sinh(\lambda_{ij}^{\frac{1}{2}}x)(\xi, \phi_{ij})\phi_{ij}$$

accordingly for $\cosh(A_i^{\frac{1}{2}}(x))$. We use the asymptotic expansions

$$\begin{cases} \sinh(A_i^{\frac{1}{2}}(\sigma^i(\rho_i)))\xi = & \sigma^i(\rho_i)A_i^{\frac{1}{2}}\xi + O(\rho_i^2) \\ \cosh(A_i^{\frac{1}{2}}(\sigma^i(\rho_i)))\xi = & \xi + O(\rho_i^2) \end{cases} \tag{35}$$

By (30) we have

$$a_i^\rho = (\sin(A_i^{\frac{1}{2}}(\ell_i))^{-1}(u_i - \cosh(A_i^{\frac{1}{2}}(\ell_i))b_i^\rho),\ i = 1, \dots, q \tag{36}$$

We expand (31) and(32)

$$\begin{aligned} A_i^{\frac{1}{2}}\rho_i a_i^\rho + b_i^\rho &= b_{q+i}^\rho \\ &= \sigma^{q+i-1}(\rho_{q+i-1})A_{q+i-1}^{\frac{1}{2}} a_{q+i-1}^\rho + b_{q+i-1}^\rho + O(\rho^2),\ i = 2, \dots q \end{aligned} \tag{37}$$

$$A_1^{\frac{1}{2}}\rho_1 a_1^\rho + b_1^\rho = b_{q+1}^\rho = \sigma^{2q}(\rho_{2q})A_{2q}^{\frac{1}{2}} a_{2q}^\rho + b_{2q}^\rho + O(\rho^2) \tag{38}$$

We now proceed to the Kirchhoff conditions at the multiple nodes (33),(34)

$$\begin{aligned} &-\tfrac{1}{c_i}A_i^{-\frac{1}{2}}[a_i^\rho + \rho_i A_i^{\frac{1}{2}} b_i^\rho] - \tfrac{1}{c_{q+i}}A_{q+i}^{-\frac{1}{2}} a_{q+i}^\rho \\ &\quad + \tfrac{1}{c_{q+i-1}}A_{q+i-1}^{-\frac{1}{2}}[a_{q+i-1}^\rho + \sigma^{q+i-1}(\rho_{q+i-1})A_{q+i-1}^{\frac{1}{2}} b_{q+i-1}^\rho] \\ &\qquad = 0 + O(\rho^2),\ i = 2, \dots, q \end{aligned} \tag{39}$$

and

$$\begin{aligned} &-\tfrac{1}{c_1}A_1^{-\frac{1}{2}}[a_1^\rho + \rho_1 A_1^{\frac{1}{2}} b_1^\rho] - \tfrac{1}{c_{q+1}}A_{q+1}^{-\frac{1}{2}} a_{q+1}^\rho \\ &\quad + \tfrac{1}{c_{2q}}A_{2q}^{-\frac{1}{2}}[a_{2q}^\rho + \sigma^{2q}(\rho_{2q})A_2^{\frac{1}{2}} b_{2q}^\rho] = 0 + O(\rho^2) \end{aligned} \tag{40}$$

We reformulate the system (37),(38),(39),(40) as follows

$$\begin{aligned} &\left[A_{i-1}^{\frac{1}{2}}\rho_{i-1} - \tanh(A_{i-1}^{\frac{1}{2}}\ell_{i-1})\right] a_{i-1}^\rho - \left[A_i^{\frac{1}{2}}\rho_i - \tanh(A_i^{\frac{1}{2}}\ell_i)\right] a_i^\rho \\ &\quad + \sigma^{q+i-1}(\rho_{q+i-1})A_{q+i-1}^{\frac{1}{2}} a_{q+i-1}^\rho \\ &\qquad = \cosh(A_i^{\frac{1}{2}}\ell_i)^{-1}u_i - \cosh(A_{i-1}^{\frac{1}{2}}\ell_{i-1})^{-1}u_{i-1},\ i = 2, \dots, q \\ &-\left[A_1^{\frac{1}{2}}\rho_1 - \tanh(A_1^{\frac{1}{2}}\ell_1)\right] a_1^\rho + \left[A_q^{\frac{1}{2}}\rho_q - \tanh(A_q^{\frac{1}{2}}\ell_q)\right] a_q^\rho \\ &\quad + \sigma^{2q}(\rho_{2q})A_{2q}^{\frac{1}{2}} a_{2q}^\rho = \cosh(A_1^{\frac{1}{2}}\ell_1)^{-1}u_1 - \cosh(A_q^{\frac{1}{2}}\ell_q)^{-1}u_q + O(\rho^2) \end{aligned} \tag{41}$$

$$\begin{aligned} &-\left[\tfrac{1}{c_i}A_i^{-\frac{1}{2}} + \left(\tfrac{\sigma^{q+i-1}(\rho_{q+i-1})}{c_{q+i-1}} - \tfrac{\rho_i}{c_i}\right)\tanh(A_i^{\frac{1}{2}}\ell_i)\right] a_i^\rho \\ &\quad - \tfrac{1}{c_{q+i}}A_{q+i}^{-\frac{1}{2}} a_{q+i}^\rho + \tfrac{1}{c_{q+i-1}}A_{q+i-1}^{-\frac{1}{2}} a_{q+i-1}^\rho \\ &\qquad = -\left(\tfrac{\sigma^{q+i-1}(\rho_{q+i-1})}{c_{q+i-1}} - \tfrac{\rho_i}{c_i}\right)\cosh(A_i^{\frac{1}{2}}\ell_i)^{-1}u_i,\ i = 2, \dots q \\ &-\left[\tfrac{1}{c_1}A_1^{-\frac{1}{2}} + \left(\tfrac{\sigma^{2q}(\rho_{2q})}{c_{2q}} - \tfrac{\rho_1}{c_1}\right)\tanh(A_1^{\frac{1}{2}}\ell_1)\right] a_1^\rho \\ &\quad - \tfrac{1}{c_{q+1}}A_{q+1}^{-\frac{1}{2}} a_{q+1}^\rho + \tfrac{1}{c_{2q}}A_{2q}^{-\frac{1}{2}} a_{2q}^\rho \\ &\qquad = -\left(\tfrac{\sigma^{2q}(\rho_{2q})}{c_{2q}} - \tfrac{\rho_1}{c_1}\right)\cosh(A_1^{\frac{1}{2}}\ell_1)^{-1}u_1 + O(\rho^2) \end{aligned} \tag{42}$$

Now, (41)-(42) constitute a system of $2q$ linear asymptotic equations to order 2 in the $2q$ variables $a_i^\rho,\ i = 1, \dots, 2q$.

Theorem 4.1 *The system of equations* (39) *to* (42) *admits a unique solution* a_i^ρ, $i = 1, \ldots 2q$. *Moreover, we have the asymptotic expansion*

$$a_i^\rho = a_i + O(\rho), \quad i = 1, \ldots, q, \tag{43}$$

where a_i *is given by* (23) *There exists a function* $s_i(\cdot)$ *such that*

$$r_i^\rho(x) = r_i(x) + O(\rho)s_i(x), \quad i = 1, \ldots, q, \tag{44}$$

where r_i *is the solution of the star-graph problem* (18) $\rho = 0$.

Proof: Using equations (37) and (38), taking appropriate differences, we realize that $b_i = \hat{b} + O(\rho)$. This information is inserted into equations (39) and (40). If we write all quantities involving a_i^ρ with indices $i = 1 \ldots q$ on the left and the other terms on the right side, we obtain after summing up, using a 'telescope-sum', only $O(\rho)$-terms on the right hand side, i.e. we have

$$\sum_{i=1}^{q} \frac{1}{c_i} A_i^{-\frac{1}{2}} a_i^\rho = O(\rho) \tag{45}$$

Then we use the expression (36) for a_i^ρ in (45) to obtain

$$\sum_{i=1}^{q} \frac{1}{c_i} A_i^{-\frac{1}{2}} \sinh(A_i^{\frac{1}{2}} \ell_i)^{-1} u_i = \left(\sum_{i=1}^{q} \frac{1}{c_i} A_i^{-\frac{1}{2}} \coth(A_i^{\frac{1}{2}} \ell_i)^{-1} \right) \hat{b}$$

From this and (24) we see that up to terms of order $O(\rho)$, $\hat{b} = b$. Then a_i^ρ, up to the order $O(\rho)$, are given by a_i in (23).

4.1 Homogeneous networks

In this subsection we consider the network under the assumption (26), i.e. all material and geometrical quantities are the same, and a symmetric hole. Under this assumption the system of equations (41) to (42) reduces to

$$\begin{array}{l} a_{i-1}^\rho - a_i^\rho - \sigma\rho\coth(\ell) a_{q+i-1}^\rho = -\frac{1+\rho\coth(\ell)}{\sinh(\ell)}(u_i - u_{i-1}) + O(\rho^2), \\ -a_1^\rho + a_q^\rho - \sigma\rho\coth(\ell) a_{2q} = -\frac{1+\rho\coth(\ell)}{\sinh(\ell)}(u_1 - u_q) + O(\rho^2), \\ -(1 + (\sigma - 1)\rho\tanh(\ell)) a_i^\rho - a_{q+i}^\rho + a_{q+i-1}^\rho = \frac{1-\sigma}{\cosh(\ell)} u_i + O(\rho^2) \\ -(1 + (\sigma - 1)\rho\tanh(\ell)) a_1^\rho - a_{q+1}^\rho + a_{2q}^\rho = \frac{1-\sigma}{\cosh(\ell)} u_1 + O(\rho^2), \end{array} \tag{46}$$

where the first and the third equations hold for $i = 2, \ldots, q$, respectively. This system has a very particular sparse structure which reflects the adjacency structure of the graph. To obtain the direct explicit solution is, nevertheless, a matter of substantial calculations. Instead we look at an example.

Example 4.2 *In this example we reduce the graph to a tripod. See figure 4. Here we can solve* (46) *analytically an obtain*

$$
\begin{aligned}
a_i^\rho &= \tfrac{1}{\sinh(\ell)}(u_i - \tfrac{1}{3}\sum_{j=1}^{3} u_j) \\
&\quad + \rho\tfrac{1}{\cosh(\ell)}\Big\{(1-\tfrac{1}{3}\sigma)\coth(\ell)^2(u_i - \tfrac{1}{3}\sum_{j=1}^{3} u_j) \\
&\qquad + (\sigma-1)\tfrac{1}{3}\sum_{j=1}^{3} u_j\Big\} + O(\rho^2),
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
b_i^\rho &= \tfrac{1}{\cosh(\ell)}\tfrac{1}{3}\sum_{j=1}^{3} u_j \\
&\quad - \rho\tfrac{\sinh(\ell)}{\cosh(\ell)^2}\Big\{\big((1-\tfrac{1}{3}\sigma)\coth(\ell)^2\big)(u_i - \tfrac{1}{3}\sum_{j=1}^{3} u_j) \\
&\qquad + (\sigma-1)\tfrac{1}{3}\sum_{i=1}^{3} u_i\Big\} + O(\rho^2),
\end{aligned} \tag{48}
$$

where $i = 1, 2, 3$.

We also display the coefficients a_{q+i}^ρ, $i = 1, 2, 3$ *in order to reveal the behavior of the edges introduced by cutting the hole.*

$$
\begin{aligned}
a_4^\rho &= \frac{1}{3\sinh(\ell)}(u_2 - u_1) \\
&\quad + \frac{\rho}{3\sinh(\ell)}\big((1-\frac{\sigma}{3})\coth(\ell))(u_2-u_1)\big) + O(\rho^2)
\end{aligned} \tag{49}
$$

$$
\begin{aligned}
a_5^\rho &= \frac{1}{3\sinh(\ell)}(u_3 - u_2) \\
&\quad + \frac{\rho}{3\sinh(\ell)}\big((1-\frac{\sigma}{3})\coth(\ell))(u_3-u_2)\big) + O(\rho^2)
\end{aligned} \tag{50}
$$

$$
\begin{aligned}
a_6^\rho &= \frac{1}{3\sinh(\ell)}(u_1 - u_3) \\
&\quad + \frac{\rho}{3\sinh(\ell)}\big((1-\frac{\sigma}{3})\coth(\ell)(u_1-u_3)\big) + O(\rho^2)
\end{aligned} \tag{51}
$$

The remaining b_{q+i}, $1 = 1, 2, 3$ *are of course given by* b_i, $i = 1, 2, 3$ *according to* (37),(38). *This completely determines the solution* $r_i^\rho(x)$, $i = 1, \ldots, 6$. *We list the first three members for easier reference:*

$$
\begin{aligned}
r_i^\rho(x) &= \tfrac{1}{\sinh(\ell)}\big(u_i - \tfrac{1}{3}\sum_{j=1}^{3} u_j\big)\sinh(x) + \tfrac{1}{\cosh(\ell)}\tfrac{1}{3}\sum_{j=1}^{3} u_j \cosh(x) \\
&\quad + \rho\Big\{\tfrac{1}{\cosh(\ell)}\Big[(1-\tfrac{1}{3}\sigma)\coth(\ell)^2(u_i - \tfrac{1}{3}\sum_{j=1}^{3} u_j) \\
&\qquad + (\sigma-1)\tfrac{1}{3}\sum_{j=1}^{3} u_j\Big]\sinh(x) \\
&\quad - \tfrac{\sinh(\ell)}{\cosh(\ell)^2}\Big[(1-\tfrac{1}{3}\sigma)\coth(\ell)^2(u_i - \tfrac{1}{3}\sum_{j=1}^{3} u_j) \\
&\quad + (\sigma-1)\tfrac{1}{3}\sum_{j=1}^{3} u_j\Big]\cosh(x)\Big\} \quad + O(\rho^2),\ i = 1, 2, 3
\end{aligned} \tag{52}
$$

The Steklov- Poincaré-map is then obtained using

$$(r_i')^\rho(\ell) = \coth(\ell)(u_i - \tfrac{1}{3}\sum_{j=1}^{3} u_j) + \tanh(\ell)\tfrac{1}{3}\sum_{j=1}^{3} u_j \\ +\rho\left\{(1-\tanh^2(\ell))[(1-\tfrac{1}{3}\sigma)\coth^2(\ell)(u_i - \tfrac{1}{3}\sum_{j=1}^{3} u_j)) \\ +(\sigma-1)\tfrac{1}{3}\sum_{j=1}^{3} u_j]\right\},\ i=1,\dots q. \tag{53}$$

It is apparent that (52),(53) provide the second order asymptotic expansion we were looking for. We consider the following experiment: we apply longitudinal forces $u_i = ue_i$ *with the same magnitude at the simple nodes of the network. The (outer) edges* $e_i,\ 1=1,2,3$ *or, respectively the edges of the original star, are given by*

$$e_1 = (0,1),\ e_2 = (-\frac{\sqrt{3}}{2}, -\frac{1}{2}),\ e_3 = (\frac{\sqrt{3}}{2}, -\frac{1}{2})$$

which together with the orthogonal complements

$$e_1^\perp = (-1,0),\ e_2^\perp = (\frac{1}{2}, -\frac{\sqrt{3}}{2}),\ e_3^\perp = (\frac{1}{2}, \frac{\sqrt{3}}{2})$$

form the local coordinate systems of the edges. Obviously $\sum_{i=1}^{3} e_i = 0$. *Thus the solution to the unperturbed problem is given by*

$$r_i(x) = \frac{1}{\sinh(\ell)} u\sinh(x)e_i \tag{54}$$

This is in agreement with the fact that that particular reference configuration is completely symmetric. Now, the solution r_i^ρ *to the perturbed system and* $(r_i')^\rho(\ell)$ *are then given by*

$$\begin{aligned} r_i^\rho(x) &= \frac{1}{\sinh(\ell)}\sinh(x)ue_i \\ &\quad +\rho(1-\frac{\sigma}{3})\frac{1}{\sinh(\ell)^2}\left(\coth(\ell)\sinh(x)-\cosh(x)\right)ue_i\ +O(\rho^2) \qquad (55) \\ (r_i)'^\rho(\ell) &= \coth(\ell)u + \rho\frac{1}{\sinh(\ell)^2}(1-\frac{\sigma}{3})ue_i\ +O(\rho^2) \end{aligned}$$

The energy of the unperturbed system is given by

$$\mathcal{E}_0 = \frac{1}{2}\sum_{i=1}^{3}\int_0^\ell r_i'\cdot r_i' + r_i\cdot r_i dx = \frac{3}{2}\coth(\ell)u^2 \tag{56}$$

The energy of the perturbed system is given by

$$\mathcal{E}^\rho = \frac{1}{2}\sum_{i=1}^{3}\int_0^{\ell-\rho}[r_i'\cdot r_i' + r_i\cdot r_i]dx + \frac{1}{2}\sum_{i=4}^{6}\int_0^{\sigma\rho}[r_i'\cdot r_i' + r_i\cdot r_i]dx \tag{57}$$

$$= \langle S^\rho u, u\rangle = \langle S^0 u, u\rangle + \rho\frac{1}{2}(1-\frac{\sigma}{3})\left\{(1-(\tanh(\ell))^2)\right\}u^2 \tag{58}$$

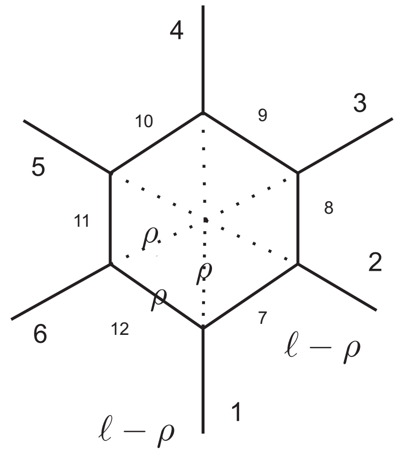

Figure 5: Graph with 'critical' edge degree 6

From these experiments we may draw the conclusion, that nodes of edge degree 3 under symmetric load, where the configuration is at 120 between the edges (this amounts to $\sigma = \sqrt{3}$*) are not going to be replaced by hole, which would, in turn result in 3 new multiple nodes of edge degree 3. This seems to support the optimality of such graphs being observed by Buttazzo [3].*

Remark:

1. *Very similar formulae are obtained in the scalar case (*$r_i(x) \in \mathbf{R}$*, no planar representation!), relevant for instance in problems of heat transfer or electrical currents in networks.*

2. *If the loads are not symmetric, and/or if the geometry of the 'hole' is not uniform, the energy may in fact drop. A more detailed analysis is subject of forthcoming paper. Suffice it to say here, that nodes with higher edge degree, according to our analysis, are 'more likely' to be released by a hole, as even in the symmetric case the number* $\sigma(\rho)$ *which measures the new edge-lengths will be less than 1.*

   This is true e.g. for a node with edge degree 6 and beyond. Thus, the total length of the new edges is smaller than the total length of the removed edges. This, in turn, is intuitive with respect to the fact that in the higher-dimensional problem (in 2- or 3-d, no graphs), digging a hole reduces the amount of mass.

Example 4.3 *Here we consider the homogeneous situation for a star with edge degree 6 at the multiple node. In this case* $\sigma = 1$ *for the symmetric situation. See Figure 5*

We calculate

$$
\begin{aligned}
a_1^\rho &= \frac{1}{\sinh(\ell)}(u_1 - \frac{1}{6}\sum_{j=1}^{6} u_j) \\
&\quad +\rho\frac{\cosh(\ell)}{\cosh^2(\ell)-1}\{(-u_5 - u_3 - 4u_2 - 4u_6 + 10u_1) \\
&\qquad -7(u_1 - \frac{1}{6}\sum_{j=1}^{6} u_j)\bigg\}
\end{aligned} \tag{59}
$$

Notice that the edges 2 and 6 are the 'neighboring' edges of edge 1 in the original star-graph. The other coefficients a_i^ρ, $1 = 2, \ldots, 6$ *are then obvious. For the sake of brevity, we only display e.g.* a_{12}^ρ*:*

$$
\begin{aligned}
a_{12}^\rho &= \frac{1}{12\sinh(\ell)}[5(u_1-u_6) + 3(u_2-u_5) + (u_3-u_4)] \\
&\quad -\rho\frac{\cosh(\ell)}{144(\cosh^2(\ell)-1)}[25(u_1-u_6) - 9(u_2-u_5) - 7(u_3-u_4)] \\
&\qquad +O(\rho^2)
\end{aligned} \tag{60}
$$

Again, observe that edge 12, in terms of the edges of the original graph, has direct neighbors 1 and 6, the next level is 2 and 5 and finally we have 3 and 4. One realizes a consequent scaling. Also note that $a_i^\rho = 0$ *if* u_i *are all equal. This shows that the coefficients* b_i^ρ *in that case are independent of* ρ *and thus the energy will not change for this limiting case.*

5 The topological derivative

We are now in the position to define the topological derivative of an elliptic problem on a graph.

Let G be a graph, and let $v_J \in \mathcal{J}_M$ be a multiple node with edge degree d_J. Let G_ρ be the graph obtained from G by replacing v_J with a cycle of length $\sum_{i=1}^{d_J} c_i\rho$ with vertices $v_J^1, \ldots v_J^{d_J}$ of edge degree 3 each, such that the distance from v_J to v_J^i is equal to ρ. Thus, the number n^ρ of edges of G_ρ is $n + d_J$. Let $\mathcal{J} : G \to \mathbf{R}$ be a functional on the edges of G

$$
J(G) := \sum_{i=1}^{n} \int_0^{\ell_i} F(x, r_i, r_i') \tag{61}
$$

and let

$$
J(G_\rho) := \sum_{i=1}^{n+d_J} \int_0^{\ell_i^\rho} F(x, r_i^\rho, (r_i^\rho)') \tag{62}
$$

be its extension to $G\rho$. Assume we have an asymptotic expansion as follows

$$
J(G_\rho) = J(G) + \rho\mathcal{T}(v_J) + O(\rho^2) \tag{63}
$$

then we define the topological gradient of $J(G_\rho)$ with respect to ρ for $\rho = 0$ at the vertex v_J as follows.

$$\mathcal{T}(v_J) = \lim_{\rho\to 0} \frac{J(G_\rho) - J(G)}{\rho} \tag{64}$$

We first consider the energy functional. There are five such functionals relevant for the analysis of this paper: $E^0(r)$ on the entire graph G , $E^\rho(r^\rho)$ on the entire graph with the hole G^ρ , $E_{CS}(r)$ on the graph $G \setminus \mathcal{S}^{J^0}$, where the star-graph without hole $\mathcal{S}^{J^0}$ has been cut out along edges $e_i,\ i \in \mathcal{I}_{J^0}$, $E^0_S(r; v)$ on the star-graph without hole, and $E^\rho_S(r; v)$ on the star-graph with hole. Obviously

$$E^0_S(r; u) = \langle S^0 u, u\rangle, \tag{65}$$

$$E^\rho_S(r; u) = \langle S^\rho u, u\rangle, \tag{66}$$

$$E^0(r) = E_{CS}(r) + E^0_S(r, r), \quad E^\rho(r^\rho) = E_{CS}(r^\rho) + E^\rho_S(r^\rho, r^\rho), \tag{67}$$

where it is understood that in $E^\rho_S(r^\rho, \cdot)$ and $E^0_S(r, \cdot)$ we insert $u_i = r^\rho(\ell_i)$ and $u_i = r^0(\ell_i)$, respectively. Thus

$$E^\rho(r^\rho) - E^0(r) = \langle S^\rho(\tilde r), \tilde r\rangle - \langle S^0(\tilde r), \tilde r\rangle, \tag{68}$$

where $\tilde r$ solves the problem on $G \setminus \mathcal{S}^{J^0}$ and $u_i = \tilde r_i(\ell_i), i \in \mathcal{I}_{J^0}$. Thus the asymptotic analysis of the last section carries over to the entire graph. As we have done the complete asymptotic analysis up to order 2 in the homogeneous case only, we consequently dwell on this case now,the more general case will be subject of a forthcoming publication.

5.1 Homogeneous graphs

In order to find an expression of the topological gradient in terms of the solutions r at the node v_{J^0}, the one that is cut out, we need to express the solution in terms of the data u_i.

Example 5.1 *We consider the star-graph as above with 3 edges. Obviously*

$$u_i - \frac{1}{3}\sum_{j=1}^{3} u_j = \sinh(\ell) r_i'(0),\ \frac{1}{3}\sum_{j=1}^{3} u_j = \cosh(\ell) r_i(0). \tag{69}$$

Thus using the fact that $\sum\limits_{i=1}^{3} \|u_i - \frac{1}{3}\sum\limits_{j=1}^{3} u_j\|^2 = \sum\limits_{i=1}^{3} \|u_i\|^2 - \frac{1}{3}(\|\sum\limits_{i=1}^{3}\|)^2$ *we can express the bilinear expression* $\langle \mathcal{S}^\rho(u), u\rangle$ *in terms of* $\|r^0(0)\|^2$ *and* $\|(r^0)'(0)\|^2$ *(where we omit the index* 0*) as follows*

$$\begin{aligned}\langle \mathcal{S}^\rho_i(u), u\rangle &= \langle \mathcal{S}^0_i(u), u\rangle \\ &+\rho\left\{(1 - \tfrac{1}{3}\sigma)\sum_{i=1}^{3}\|r_i'(0)\|^2 + (\sigma - 1)\sum_{i=1}^{3}\|r_i(0)\|^2\right\}\end{aligned} \tag{70}$$

This says that the energy function in the homogeneous case, when cutting out a symmetric hole e.g. $\sigma^i = \sigma = \sqrt{3},\ i = 1, 2, 3,$ *we have*

$$\mathcal{T}_E(r, v_{J^0}) = \left\{ (1 - \frac{1}{3}\sigma) \sum_{i=1}^{3} \|r_i'(0)\|^2 + (\sigma - 1) \sum_{i=1}^{3} \|r_i(0)\|^2 \right\} \tag{71}$$

The situation will be different for such vertices having a higher edge-degree as 6, and those having non-symmetric holes. We expect that such networks are more likely to be reduced to edge-degree 3 by tearing a hole. But this has to be confirmed by more detailed studies.

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